

# DOUBLE-BOSONIZATION AND MAJID'S CONJECTURE, (I): RANK-INDUCTIONS OF $ABCD$

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**ABSTRACT.** Majid developed in [M3] the double-bosonization theory to construct  $U_q(\mathfrak{g})$  and expected to generate inductively not just a line but a tree of quantum groups starting from a node. In this paper, the authors confirm the Majid's first expectation (see p. 178 [M3]) through giving and verifying the full details of the inductive constructions of  $U_q(\mathfrak{g})$  for the classical types, i.e., the  $ABCD$  series. Some examples in low ranks are given to elucidate that any quantum group of classical type can be constructed from the node corresponding to  $U_q(\mathfrak{sl}_2)$ .

## 1. INTRODUCTION

The invention of quantum groups is one of the outstanding achievements of mathematical physics and mathematics in the late twentieth century, which arose in the work of L. D. Faddeev with his school for solving integrable models in 1+1 dimension by the quantum inverse scattering method. In the history of development, a major event was the discovery of quantized universal enveloping algebras  $U_q(\mathfrak{g})$  over the complex field by V. G. Drinfeld [D] and M. Jimbo [Ji] independently around 1985. A striking feature of quantum group theory is the close connections with many branches of mathematics and physics, such as Lie groups, Lie algebras and their representations, representation theory of Hecke algebras, link invariant theory, conformal field theory, and so on. This attracts many mathematicians to find some better way in a suitable frame to understand the structure of quantum groups defined initially by generators and relations. For instance, the first we must mention is the famous FRT-construction [FRT] of  $U_q(\mathfrak{g})$  for the classical types based on the  $R$ -matrices of the vector representations associated to the classical simple Lie algebras  $\mathfrak{g}$ , which is a natural analogue of the matrix realization of the classical Lie algebras. It is a belief that the commutation relations in quantum groups expressed by means of the  $R$ -matrices are of fundamental importance for the theory. Afterwards, Majid rediscovered the so-called “Radford-Majid bosonization” arising from a framework of a braided category over a Hopf

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algebra (see [Ra], [RT], [M1], [M2], [M9]), and then, based on the FRT-construction and extending the Drinfeld double to the generalized quantum double associated to a dual pair of Hopf algebras equipped with a (not necessarily non-degenerate) pairing, Majid [M3] in 1996 developed the double-bosonization theory to give a direct construction of  $U_q(\mathfrak{g})$  through viewing the Lusztig's algebra  $\mathfrak{f}$  ([L2]) as a braided group in a special braided category  $\mathfrak{M}_H$  (or  ${}_H\mathfrak{M}$ ), where  $(H = kQ = k[K_i^{\pm 1}], A = kQ^\vee)$  is a weakly quasitriangular dual pair (see p. 169 of [M3]), where  $Q$  (resp.  $Q^\vee$ ) is a (dual) root lattice of  $\mathfrak{g}$ . An analogous construction in spirit in the Yetter-Drinfeld category of a Hopf algebra was given independently by Sommerhäuser in [So].

Besides the constructions above, in the early 90s, Ringel [Ri] realized the positive part of quantum groups by quiver representations and Hall algebras, which inspired Lusztig's canonical bases theory [L1, L2]. Rosso [Ro] also realized the positive part of  $U_q(\mathfrak{g})$  by introducing the quantum shuffle algebra in a braided category, and gave a recipe of axiomatic inductive construction based on the quantum “shuffle” operation defined on the (braided) tensor algebra of a braided vector space which is sometimes a bit inconvenient for making practical calculations, in contrast with the Majid's double-bosonization [M3]. Bridgeland [Br] recently realized the entire quantum groups of type  $ADE$  using the Ringel-Hall algebras. From another way, Fang-Rosso [FR] realized the whole quantum groups by means of a new theory on the quantum quasi-symmetric algebras due to Jian-Rosso [JR]. More general, the whole axiomatic description for the multi-parameter quantum groups as well as the Hopf 2-cocycle deformation under the machinery of multi-parameter quasi-symmetric algebras have been achieved in Hu-Li-Rosso [HLR].

In this paper, let us focus on the double-bosonization theory in [M3].

Associated to any mutually dual braided groups  $B^\star, B$  covariant under a background quasitriangular Hopf algebra  $H$ , there is a new quantum group on the tensor space  $B^\star \otimes H \otimes B$  by double-bosonization in [M3], consisting of  $H$  extended by  $B$  as additional ‘positive roots’ and its dual  $B^\star$  as additional ‘negative roots’. The construction is more powerful than the quantum double since one can reach directly to the  $U_q(\mathfrak{g})$  (the quantum double is a bit big and one has to make quotient). Specially, Majid viewed  $U_q(\mathfrak{n}^\pm)$  as the mutually dual braided groups in braided category of right “Cartan subalgebra”  $H$ -modules, then recovered  $U_q(\mathfrak{g})$  by his double-bosonization theory. Also, based on two examples in low ranks given in [M3, M8], Majid claimed that many new quantum groups, as well as the inductive (by rank) construction of  $U_q(\mathfrak{g})$  can be obtained in principle by this theory [M3] (we call it the Majid's expectation), since he imaged his double-bosonization framework allows to generate a tree of quantum groups and at each node of the tree, there are many choices to adjoin a pair of braided groups covariant under the corresponding quantum group at that

node. Actually, it is a combinatorial and representation-theoretical challenge to elaborate the full tree structure, which brings us the first motivation. We want to describe in details the inductive construction of quantum groups  $U_q(\mathfrak{g})$  for all complex semisimple Lie algebras  $\mathfrak{g}$ , which just elaborates some main branches of the tree. We will limit ourself to consider the classical  $ABCD$  series in this paper, which is organized as follows.

In section 2, we recall some basic facts about the FRT-construction and the Majid's double-bosonization construction. In section 3, we analyse explicitly the procedure of the inductive construction of  $U_q(\mathfrak{sl}_n)$ , and find some tips in low rank cases. That is, although the entire  $FRT$ -matrix  $m^\pm$  is hard to know from the vector representation, we only need to know those diagonal and minor diagonal elements. This inspires us how to proceed the general rank-inductive construction from  $U_q(\mathfrak{sl}_n)$  to  $U_q(\mathfrak{sl}_{n+1})$ . At the node diagram of  $U_q(\mathfrak{sl}_n)$ , we can choose a pair of braided groups corresponding to the standard  $R$ -matrix arising from the vector representation. Similarly, we can do the same thing for the  $BCD$  series. These demonstrate that the Majid's double-bosonization framework does allow to generate four branching-lines of nodes diagram of quantum groups. Furthermore, in the last section, we will give some examples to show how to grow the tree (with  $ABCD$ -branches) of nodes diagram of quantum groups out of the same 'root' node at type  $A$  via type-crossing construction starting from type  $A_r$  for  $r = 1, 2, 3$ .

## 2. FRT-CONSTRUCTION AND MAJID'S DOUBLE-BOSONIZATION CONSTRUCTION

In this paper, let  $k$  be the complex field,  $\mathfrak{g}$  a finite-dimensional complex simple Lie algebra with simple roots  $\alpha_i$ . Let  $\lambda_i$  be the fundamental weight corresponding to  $\alpha_i$ . Cartan matrix of  $\mathfrak{g}$  is  $(a_{ij})$ , where  $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ , and  $d_i = \frac{(\alpha_i, \alpha_i)}{2}$ . Let  $(H, \mathcal{R})$  be a quasitriangular Hopf algebra, where  $\mathcal{R}$  is the universal  $R$ -matrix,  $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ ,  $\mathcal{R}_{21} = \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)}$ . Denote by  $\Delta, \eta, \epsilon, S$  its coproduct, counit, unit, its antipode, respectively. We shall use Sweedler's notation: for  $h \in H$ ,  $\Delta(h) = h_1 \otimes h_2$ . Write  $H^{op}$  ( $H^{cop}$ ) the opposite (co)algebra structure of  $H$ , respectively. Let  $\mathfrak{M}_H$  ( ${}_H\mathfrak{M}$ ) be the braided category consisting of right (left)  $H$ -modules, respectively. If there exists a coquasitriangular Hopf algebra  $A$  such that  $(H, A)$  is a weakly quasitriangular dual pair, then  $\mathfrak{M}_H$  ( ${}_H\mathfrak{M}$ ) is equivalent to the braided category  ${}^A\mathfrak{M}$  ( $\mathfrak{M}^A$ ) consisting of left (right)  $A$ -comodules, respectively. For the detailed description of these theories, we left to the readers to refer to Drinfeld's and Majid's papers [D], [M4], [M5], and so on. By a braided group, following [M4], we mean a braided bialgebra or Hopf algebra in some braided category. In order to distinguish from the ordinary Hopf algebras, denote by  $\underline{\Delta}, \underline{S}$  its coproduct and antipode, respectively. An invertible matrix solution of the quantum Yang-Baxter equation (QYBE)  $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$  is called a  $R$ -matrix. For later use, one needs a certain extension  $U_q^{ext}(\mathfrak{g})$  of  $U_q(\mathfrak{g})$ . This is constructed by adjoining

formally certain products of the elements  $K_i^{\pm \frac{1}{n}}$ ,  $i = 1, \dots, n-1$  to  $U_q(\mathfrak{sl}_n)$ , the elements  $K_n^{\pm \frac{1}{2}}$  to  $U_q(\mathfrak{sp}_{2n})$ , and  $K_{n-1}^{\pm \frac{1}{2}} K_n^{\pm \frac{1}{2}}$  and  $K_{n-1}^{\pm \frac{1}{2}} K_n^{\mp \frac{1}{2}}$  to  $U_q(\mathfrak{so}_{2n})$ . The precise definitions of them can be found in [KS]. Denote by  $T_V$  an irreducible representation of  $U_q(\mathfrak{g})$ , where  $V$  is the corresponding module with a basis  $\{x_i\}$ .

**2.1. FRT-construction.** One can obtain an invertible (basic)  $R$ -matrix from the quasitriangular Hopf algebra  $U_q(\mathfrak{g})$  and its (vector) representation. Conversely, starting from an invertible (basic)  $R$ -matrix, if does there exist a quasitriangular Hopf algebra to recover such a  $R$ -matrix through a suitable representation? First of all, by Faddeev-Reshetikhin-Takhtajan [FRT], the following fact is basic and well-known.

**Definition 2.1.** *Given an invertible matrix solution  $R$  of the QYBE, there is a bialgebra  $A(R)$ , named the FRT-bialgebra, which is generated by 1 and  $t_j^i$ , for  $1 \leq i, j \leq n$ , with the relations  $RT_1T_2 = T_2T_1R$ ,  $\Delta(T) = T \otimes T$  and  $\epsilon(T) = I$  using standard notation in [FRT], where the matrix  $T = (t_j^i)$ ,  $T_1 = T \otimes I$ ,  $T_2 = I \otimes T$ .*

Observe that  $A(R)$  is a coquasitriangular bialgebra with  $\mathcal{R} : A(R) \otimes A(R) \rightarrow k$  such that  $\mathcal{R}(t_j^i \otimes t_l^k) = R_{jl}^{ik}$ . Here  $R_{jl}^{ik}$  denotes the entry at row  $(ik)$  and column  $(jl)$  in matrix  $R$ . Secondly, in the dual space  $A(R)^* = \text{Hom}(A(R), k)$ ,  $\Delta$  of  $A(R)$  induces the multiplication of  $A(R)^*$ . In [FRT],  $U_R$  is defined to be the subalgebra of  $A(R)^*$  generated by  $L^\pm = (l_{ij}^\pm)$ , with relations

$$(PRP)L_1^\pm L_2^\pm = L_2^\pm L_1^\pm (PRP), \quad (PRP)L_1^+ L_2^- = L_2^- L_1^+ (PRP),$$

where  $l_{ij}^\pm$  is defined by  $(l_{ij}^+, t_l^k) = R_{lj}^{ki}$ ,  $(l_{ij}^-, t_l^k) = (R^{-1})_{jl}^{ik}$ .

The algebra  $U_R$  is a bialgebra with coproduct  $\Delta(L^\pm) = L^\pm \otimes L^\pm$  and counit  $\epsilon(l_{ij}^\pm) = \delta_{ij}$ . Specially, when  $R$  is the classical  $R$ -matrix, bialgebra  $A(R)$  has a quotient coquasitriangular Hopf algebra, denoted by  $\text{Fun}(G_q)$  or  $\mathcal{O}_q(G)$ , and  $U_R$  also has a corresponding quotient quasitriangular Hopf algebra, which is isomorphic to the extended quantized enveloping algebra  $U_q^{\text{ext}}(\mathfrak{g})$ . Moreover, there exists a (non-degenerate) dual pairing  $\langle, \rangle$  between  $\mathcal{O}_q(G)$  and  $U_q^{\text{ext}}(\mathfrak{g})$ . The way of getting the resulting quasitriangular algebras  $U_q^{\text{ext}}(\mathfrak{g})$  is the so-called FRT-construction of the quantized enveloping algebras (for the classical types).

Motivated by the work of [FRT], Majid built the theory of the weakly quasitriangular dual pairings associated with  $R$ -matrices [M3] in a more general context.

**Remark 2.1.** *The  $R$ -matrices used in Majid's papers [M1, M3] are a bit different from the standard ones as in [FRT], which are the conjugations  $P \circ \cdot \circ P$  of the ordinary  $R$ -matrices by the permutation matrix  $P : P(u \otimes v) = v \otimes u$ . It can be checked directly that  $A(P \circ R \circ P) = A(R)^{\text{op}}$ , where  $(P \circ R \circ P)_{kl}^{ij} = R_{lk}^{ji}$ . Note that we will use Majid's notation for  $R$ -matrices as in [M3] in the remaining sections of this paper.*

**2.2. Majid's double-bosonization.** Majid [M3] proposed the concept of a weakly quasi-triangular dual pair via his insight on more examples on matched pairs of bialgebras or Hopf algebras in [M7]. This allowed him to establish a theory of double-bosonization in a broad framework that generalized the FRT's construction which was limited to the classical types.

**Definition 2.2.** Let  $(H, A)$  be a pair of Hopf algebras equipped with a dual pairing  $\langle, \rangle$  and convolution-invertible algebra/anti-coalgebra maps  $\mathcal{R}, \bar{\mathcal{R}} : A \rightarrow H$  obeying

$$\langle \bar{\mathcal{R}}(a), b \rangle = \langle \mathcal{R}^{-1}(b), a \rangle, \quad \partial^R h = \mathcal{R} * (\partial^L h) * \mathcal{R}^{-1}, \quad \partial^R h = \bar{\mathcal{R}} * (\partial^L h) * \bar{\mathcal{R}}^{-1}$$

for  $a, b \in A, h \in H$ . Here  $*$  is the convolution product in  $\text{hom}(A, H)$  and  $(\partial^L h)(a) = \langle h_{(1)}, a \rangle h_{(2)}$ ,  $(\partial^R h)(a) = h_{(1)} \langle h_{(2)}, a \rangle$  are left, right “differentiation operators” regarded as maps  $A \rightarrow H$  for fixed  $h$ .

Let  $C, B$  be a pair of braided groups in  $\mathfrak{M}_H$ , which are called dually paired if there is an intertwiner  $ev : C \otimes B \rightarrow k$  such that  $ev(cd, b) = ev(d, b_{(1)})ev(c, b_{(2)})$ ,  $ev(c, ab) = ev(c_{(2)}, a)ev(c_{(1)}, b)$ ,  $\forall a, b \in B, c, d \in C$ . Then  $C^{op/cop}$  (with opposite product and coproduct) is a Hopf algebra in  ${}_H\mathfrak{M}$ , which is dual to  $B$  in the sense of an ordinary dual pairing  $\langle, \rangle$  with  $H$ -bicovariant:  $\langle h \triangleright c, b \rangle = \langle c, b \triangleleft h \rangle$  for all  $h \in H$ . Let  $\bar{C} = (C^{op/cop})^{cop}$ , then  $\bar{C}$  is a braided group in  ${}_{\bar{H}}\mathfrak{M}$ , where  $\bar{H}$  is  $(H, \mathcal{R}_{21}^{-1})$ . With these, Majid gave the following double-bosonization theorem.

**Theorem 2.1. (Majid)** On the tensor space  $\bar{C} \otimes H \otimes B$ , there is a unique Hopf algebra structure  $U = U(\bar{C}, H, B)$  such that  $H \ltimes B$  (bosonization) and  $\bar{C} \ltimes H$  (bosonization) are sub-Hopf algebras by the canonical inclusions, with cross relation

$$bh = h_{(1)}(b \triangleleft h_{(2)}), \quad ch = h_{(2)}(c \triangleleft h_{(1)}), \quad (C1)$$

here  $b \in B, c \in C, h \in H$ . If there exists a coquasitriangular Hopf algebra  $A$  such that  $(H, A)$  is a weakly quasitriangular dual pair, and  $b, c$  are primitive elements, then some relations simplify to

$$[b, c] = \mathcal{R}(\bar{b}^{(1)}) \langle c, \bar{b}^{(2)} \rangle - \langle \bar{c}^{(1)}, b \rangle \bar{\mathcal{R}}(\bar{c}^{(2)}); \quad (C2)$$

$$\Delta b = \bar{b}^{(2)} \otimes \mathcal{R}(\bar{b}^{(1)}) + 1 \otimes b, \quad \Delta c = c \otimes 1 + \bar{\mathcal{R}}(\bar{c}^{(2)}) \otimes \bar{c}^{(1)}. \quad (C3)$$

Let  $\widetilde{U(\bar{R})}$  be the double cross product bialgebra of  $A(R)^{op}$  in [M7] generated by  $m^\pm$  with the bialgebra structure given by

$$Rm_1^\pm m_2^\pm = m_2^\pm m_1^\pm R, \quad Rm_1^+ m_2^- = m_2^- m_1^+ R, \quad \Delta((m^\pm)_j^i) = (m^\pm)_j^a \otimes (m^\pm)_a^i, \quad \epsilon((m^\pm)_j^i) = \delta_{ij}, \quad (C4)$$

where  $(m^\pm)_j^i$  are the *FRT*-generators,  $m^\pm$  are called *FRT*-matrices.

**Proposition 2.1.** (1)  $(\widetilde{U(R)}, A(R))$  is a weakly quasitriangular dual pair with

$$\langle (m^+)_j, t_l^k \rangle = R_{jl}^{ik}, \quad \langle (m^-)_j, t_l^k \rangle = (R^{-1})_{lj}^{ki}, \quad \mathcal{R}(T) = m^+, \quad \bar{\mathcal{R}}(T) = m^-.$$

Specially, when  $R$  is the classical  $R$ -matrix, then the weakly quasitriangular dual pair can be descended to a mutually dual pair of quotient Hopf algebras  $(U_q^{ext}(\mathfrak{g}), O_q(G))$  (where  $G$  is one of connected and simply connected algebraic groups of classical types) with the correct modification [M3]:

$$\langle (m^+)_j, t_l^k \rangle = \lambda R_{jl}^{ik}, \quad \langle (m^-)_j, t_l^k \rangle = \lambda^{-1} (R^{-1})_{lj}^{ki}.$$

Such  $\lambda$  is called a quantum group normalization constant.

(2) Suppose that  $R'$  is another matrix such that (i)  $R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12}$ ,  $R_{23}R_{13}R'_{12} = R'_{12}R_{13}R_{23}$ , (ii)  $(PR+1)(PR'-1) = 0$ , (iii)  $R_{21}R'_{12} = R'_{21}R_{12}$ , where  $P$  is the permutation matrix with entries  $P_{kl}^{ij} = \delta_{il}\delta_{jk}$ . Then a braided-vector algebra  $V(R', R)$  generated by generators  $1, \{e^i \mid i = 1, \dots, n\}$ , and relations  $e^i e^j = \sum_{a,b} R_{ab}^{ji} e^a e^b$  forms a braided group with  $\underline{\Delta}(e^i) = e^i \otimes 1 + 1 \otimes e^i$ ,  $\underline{\epsilon}(e^i) = 0$ ,  $\underline{S}(e^i) = -e^i$ ,  $\Psi(e^i \otimes e^j) = \sum_{a,b} R_{ab}^{ji} e^a \otimes e^b$  in braided category  ${}^{A(R)}\mathfrak{M}$ . Under duality  $\langle f_j, e^i \rangle = \delta_{ij}$ , a braided-covector algebra  $V^\vee(R', R_{21}^{-1})$  generated by  $1$  and  $\{f_j \mid j = 1, \dots, n\}$ , and relations  $f_i f_j = \sum_{a,b} f_b f_a R_{ab}^{ij}$  forms another braided group with  $\underline{\Delta}(f_i) = f_i \otimes 1 + 1 \otimes f_i$ ,  $\underline{\epsilon}(f_i) = 0$ ,  $\underline{S}(f_i) = -f_i$ ,  $\Psi(f_i \otimes f_j) = \sum_{a,b} f_b \otimes f_a R_{ij}^{ab}$  in braided category  $\mathfrak{M}^{A(R)}$ .

**Remark 2.2.** In fact, by Remark 2.1,  $\widetilde{U(R)}$  determined uniquely by relations (C4) with the  $R$ -matrix in [M3] is the opposite of  $U_R$  constructed by FRT-approach. So the quantized enveloping algebra  $U_q(\mathfrak{g})$  recovered by Majid (Proposition 4.3 in [M3]) satisfies  $E_i K_j = q_i^{a_{ij}} K_j E_i$ ,  $F_i K_j = q_i^{-a_{ij}} K_j F_i$ ,  $[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$ , where  $q_i = q^{d_i}$ , which is the opposite of the ordinary  $U_q(\mathfrak{g})$ . By abuse of notation, we still use  $U_q(\mathfrak{g})$  instead of  $U_q(\mathfrak{g})^{op}$  in the remaining sections of this paper.

With the new  $\lambda R$ -matrix in **Proposition 2.1**, in order that  $V(R', R), V^\vee(R', R_{21}^{-1})$  are still braided groups, then  $(U_q^{ext}(\mathfrak{g}), O_q(G))$  must be centrally extended to the pair

$$(\widetilde{U_q^{ext}(\mathfrak{g})} = U_q(\mathfrak{g}) \otimes k[c, c^{-1}], \widetilde{O_q(G)} = O_q(G) \otimes k[g, g^{-1}])$$

with action  $e^i \triangleleft c = \lambda e^i$ ,  $f_i \triangleleft c = \lambda f_i$ ,  $\langle c, g \rangle = \lambda$ . By Theorem 2.1, we have the following

**Corollary 2.1.**  $U = U(V^\vee(R', R_{21}^{-1}), \widetilde{U_q^{ext}(\mathfrak{g})}, V(R', R))$  is a new quantum group with the cross relations:  $e^i (m^+)_k^j = \lambda R_{ab}^{ji} (m^+)_k^a e^b$ ,  $(m^-)_j^i e^k = \lambda R_{ab}^{ki} e^a (m^-)_j^b$ ,  $(m^+)_j^i f_k = \lambda f_b (m^+)_a^i R_{jk}^{ab}$ ,  $f_i (m^-)_k^j = \lambda (m^-)_b^j f_a R_{ik}^{ab}$ ,  $c f_i = \lambda f_i c$ ,  $e^i c = \lambda c e^i$ ,  $[c, m^\pm] = 0$ ,  $[e^i, f_j] = \delta_{ij} \frac{(m^+)_j^i c^{-1} - c (m^-)_j^i}{q_* - q_*^{-1}}$ ;

and the coproduct:  $\Delta c = c \otimes c$ ,  $\Delta e^i = e^a \otimes (m^+)_a^i c^{-1} + 1 \otimes e^i$ ,  $\Delta f_i = f_i \otimes 1 + c(m^-)_i^a \otimes f_a$ ,  $\epsilon e^i = \epsilon f_i = 0$ , where one can normalize  $e^i$  such that the factor  $q_* - q_*^{-1}$  satisfies the situation you need.

So the Majid's double-bosonization construction can lead to new quantum groups. After giving an example of obtaining  $U_q(\mathfrak{sl}_3)$  from  $U_q^{ext}(\mathfrak{sl}_2)$  in [M3], Majid expected that the novel resulting quantum group is the quantum group of higher-one rank in the classical  $ABCD$ 's series. In the current paper, we will solve such Majid's expectation in the classical types that has been an open question aimed by Majid ([M3]) since the mid of 90's. In the next section, we will give full details of the rank-inductive construction in the  $ABCD$  series.

### 3. RANK-INDUCTIVE CONSTRUCTION OF QUANTUM GROUPS FOR CLASSICAL TYPES

In order to explore the structure of the resulting quantum group in **Corollary 2.1**, we need to know how to get the explicit form of the  $FRT$ -matrix  $m^\pm$ , which can be obtained by the following lemma.

**Lemma 3.1.** *Corresponding to the invertible matrix  $R$  obeying the QYBE, we have  $S(l_{ij}^\pm) = (m^\pm)_j^i$  in the quotient Hopf algebra,  $S$  is the corresponding antipode.*

*Proof.* If there exists a quotient Hopf algebra of  $\widehat{U(R)}$ , according to **Remark 2.2**, we obtain the following relations in this quotient Hopf algebra:  $RL_2^\pm L_1^\pm = L_1^\pm L_2^\pm R$ ,  $RL_2^- L_1^+ = L_1^+ L_2^- R$ , and  $\Delta(L^\pm) = L^\pm \otimes L^\pm$ . We can describe these relations in view of the entries in the matrix  $R$  and  $L^\pm$ . For example, for any fixed  $i, j, k, l$ , we have  $(RL_2^\pm L_1^\pm)_{kl}^{ij} = (L_1^\pm L_2^\pm R)_{kl}^{ij}$ , then we obtain the following equality on the left hand side

$$\begin{aligned} (RL_2^\pm L_1^\pm)_{kl}^{ij} &= R_{ab}^{ij} (I \otimes L^\pm)_{mn}^{ab} (L^\pm \otimes I)_{kl}^{mn} = R_{ab}^{ij} \delta_{am} (l^\pm)_{bn} (l^\pm)_{mk} \delta_{nl} \\ &= R_{mb}^{ij} (l^\pm)_{bl} (l^\pm)_{mk} = R_{mn}^{ij} (l^\pm)_{nl} (l^\pm)_{mk}, \end{aligned}$$

and the equality by the right hand side

$$(L_1^\pm L_2^\pm R)_{kl}^{ij} = (L^\pm \otimes I)_{ab}^{ij} (I \otimes L^\pm)_{mn}^{ab} R_{kl}^{mn} = (l^\pm)_{ia} \delta_{jb} \delta_{am} (l^\pm)_{bn} R_{kl}^{mn} = (l^\pm)_{im} (l^\pm)_{jn} R_{kl}^{mn}.$$

So  $R_{mn}^{ij} (l^\pm)_{nl} (l^\pm)_{mk} = (l^\pm)_{im} (l^\pm)_{jn} R_{kl}^{mn}$ . Taking the antipode  $S$  on both sides, we obtain

$$R_{mn}^{ij} S((l^\pm)_{mk}) S((l^\pm)_{nl}) = S((l^\pm)_{jn}) S((l^\pm)_{im}) R_{kl}^{mn}.$$

Under the notation  $(m^\pm)_j^i = S(l_{ij}^\pm)$ , we have

$$R_{mn}^{ij} (m^\pm)_k^m (m^\pm)_l^n = (m^\pm)_n^j (m^\pm)_m^i R_{kl}^{mn}, \text{ i.e., } R m_1^\pm m_2^\pm = m_2^\pm m_1^\pm R.$$



We also obtain  $Rm_1^+m_2^- = m_2^-m_1^+R$  by a similar argument. On the other hand,

$$\begin{aligned}\Delta((m^\pm)_j^i) &= \Delta(S(l_{ij}^\pm)) = P \circ (S \otimes S)\Delta(l_{ij}^\pm) = P \circ (S \otimes S)(l_{ia}^\pm \otimes l_{aj}^\pm) \\ &= P((m^\pm)_a^i \otimes (m^\pm)_j^a) = (m^\pm)_j^a \otimes (m^\pm)_a^i.\end{aligned}$$

So, the generators  $(m^\pm)_j^i$  satisfy relation (C4). This completes the proof.  $\square$

**3.1. Inductive construction of  $U_q(\mathfrak{sl}_n)$ .** Since  $m^\pm$  and  $R$ -matrix will become larger and larger with the growing of rank, it is a challenge to describe explicitly the procedure of general inductive construction. We want to find some tips on some concrete examples. Majid already described explicitly the case of  $U(V^\vee(R', R_{21}^{-1}), U_q^{ext}(\mathfrak{sl}_2), V(R', R)) \simeq U_q(\mathfrak{sl}_3)$  in [M3]. First of all, in order to capture much more hints coming from the theory of Majid's double-bosonization construction, in what follows, we will describe in detail an example of rank 2 case:  $U(V^\vee(R', R_{21}^{-1}), U_q^{ext}(\mathfrak{sl}_3), V(R', R)) \simeq U_q(\mathfrak{sl}_4)$ .

**Example 3.1.** *Let us start with  $R$ -matrix datum*

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & q^2 - 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & q^2 - 1 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & q^2 - 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 \end{pmatrix}.$$

Take  $R' = q^{-2}R$ , then  $R, R'$  give braided groups  $V^\vee(R', R_{21}^{-1}) = k\langle f_i \mid i = 1, 2, 3 \rangle$  and  $V(R', R) = k\langle e^i \mid i = 1, 2, 3 \rangle$ . Identify  $e^3, f_3, (m^+)_3^3 c^{-1}$  with the additional simple root vectors  $E_3, F_3$  and group-like element  $K_3$ , then the resulting quantum group  $(V^\vee(R', R_{21}^{-1}), U_q^{ext}(\mathfrak{sl}_3), V(R', R))$  is exactly the quantum group  $U_q(\mathfrak{sl}_4)$  with  $K_i^{\frac{1}{3}}, i = 1, 2$  adjoined.

*Proof.* The quantum group normalization constant for  $R$  needed for the weakly quasitriangular structure on  $U_q^{ext}(\mathfrak{sl}_3)$  is  $\lambda = q^{-\frac{4}{3}}$  obtained by the facts in [FRT]. Moreover, the  $m^\pm$ -matrix corresponding to the vector representation can be obtained by Lemma 3.1.

$$m^+ = \begin{pmatrix} K_1^{\frac{2}{3}} K_2^{\frac{1}{3}} & (q - q^{-1})E_1 K_1^{-\frac{1}{3}} K_2^{\frac{1}{3}} & q^{-1}(q - q^{-1})[E_1, E_2]_{q^{-1}} K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} \\ 0 & K_1^{-\frac{1}{3}} K_2^{\frac{1}{3}} & (q - q^{-1})E_2 K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} \\ 0 & 0 & K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} \end{pmatrix},$$



$$m^- = \begin{pmatrix} K_1^{-\frac{2}{3}} K_2^{-\frac{1}{3}} & 0 & 0 \\ (q - q^{-1}) K_1^{\frac{1}{3}} K_2^{-\frac{1}{3}} F_1 & K_1^{\frac{1}{3}} K_2^{-\frac{1}{3}} & 0 \\ q(q - q^{-1}) K_1^{\frac{1}{3}} K_2^{\frac{2}{3}} [F_2, F_1]_q & (q - q^{-1}) K_1^{\frac{1}{3}} K_2^{\frac{2}{3}} F_2 & K_1^{\frac{1}{3}} K_2^{\frac{2}{3}} \end{pmatrix},$$

where  $[E_1, E_2]_{q^{-1}} = E_1 E_2 - q^{-1} E_2 E_1$ ,  $[F_2, F_1]_q = F_2 F_1 - q F_1 F_2$ . By Corollary 2.1, we get

$$[e^3, f_3] = \frac{(m^+)_3^3 c^{-1} - c(m^-)_3^3}{q - q^{-1}}, \implies [E_3, F_3] = \frac{K_3 - K_3^{-1}}{q - q^{-1}}.$$

$E_3 K_3 = e^3 (m^+)_3^3 c^{-1} = \lambda R_{ab}^{33} (m^+)_3^a e^b c^{-1} = R_{ab}^{33} (m^+)_3^a c^{-1} e^b = R_{33}^{33} (m^+)_3^3 c^{-1} e^3 = q^2 K_3 E_3$ . From the expression of  $(m^+)_i^j$ , we have  $(m^+)_2^2 K_1 = (m^+)_1^1$ ,  $(m^+)_3^3 K_2 = (m^+)_2^2$ . Associating with the cross relation  $e^3 (m^+)_i^j = \lambda R_{ab}^{i3} (m^+)_i^a e^b$ , we obtain

$$\left. \begin{aligned} e^3 (m^+)_1^1 &= q^{-\frac{1}{3}} (m^+)_1^1 e^3, \\ e^3 (m^+)_2^2 &= q^{-\frac{1}{3}} (m^+)_2^2 e^3, \\ e^3 (m^+)_3^3 &= q^{\frac{2}{3}} (m^+)_3^3 e^3, \end{aligned} \right\} \implies \begin{cases} e^3 K_1 = K_1 e^3, \\ e^3 K_2 = q^{-1} K_2 e^3. \end{cases} \iff \begin{cases} E_3 K_1 = K_1 E_3, \\ E_3 K_2 = q^{-1} K_2 E_3. \end{cases}$$

In order to explore the relations between  $F_3$  and  $K_i, i = 1, 2, 3$ , we have  $K_3 F_3 = (m^+)_3^3 c^{-1} f_3 = \frac{1}{\lambda} (m^+)_3^3 f_3 c^{-1} = \frac{1}{\lambda} \lambda f_3 (m^+)_3^3 c^{-1} = f_3 (m^+)_3^3 c^{-1} R_{33}^{33} = q^2 F_3 K_3$  through the cross relation  $(m^+)_i^j f_3 = \lambda f_b (m^+)_a^j R_{i3}^{ab}$ . Similarly,

$$\left. \begin{aligned} (m^+)^{i+1}_{i+1} K_i &= (m^+)_i^i, \quad i = 1, 2, \\ (m^+)_i^i f_3 &= q^{-\frac{1}{3}} f_3 (m^+)_i^i, \quad i = 1, 2, \\ (m^+)_3^3 f_3 &= q^{\frac{2}{3}} f_3 (m^+)_3^3. \end{aligned} \right\} \implies \begin{cases} f_3 K_1 = K_1 f_3, \\ f_3 K_2 = q K_2 f_3. \end{cases} \iff \begin{cases} F_3 K_1 = K_1 F_3, \\ F_3 K_2 = q K_2 F_3. \end{cases}$$

Then the relations between  $K_3$  and  $E_i, F_i, i = 1, 2$  are given as follows

$$\begin{aligned} E_1 K_3 &= E_1 K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} c^{-1} = q^{-\frac{2}{3}} q^{\frac{2}{3}} K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} E_1 c^{-1} = K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} c^{-1} E_1 = K_3 E_1, \\ E_2 K_3 &= E_2 K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} c^{-1} = q^{\frac{1}{3}} q^{-\frac{4}{3}} K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} E_2 c^{-1} = q^{-1} K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} c^{-1} E_2 = q^{-1} K_3 E_2, \\ F_1 K_3 &= F_1 K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} c^{-1} = q^{\frac{2}{3}} q^{-\frac{2}{3}} K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} F_1 c^{-1} = K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} c^{-1} F_1 = K_3 F_1, \\ F_2 K_3 &= F_2 K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} c^{-1} = q^{-\frac{1}{3}} q^{\frac{4}{3}} K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} F_2 c^{-1} = q K_1^{-\frac{1}{3}} K_2^{-\frac{2}{3}} c^{-1} F_2 = q K_3 F_2. \end{aligned}$$

Moreover,  $\Delta(E_3) = E_3 \otimes K_3 + 1 \otimes E_3$ ,  $\Delta(F_3) = F_3 \otimes 1 + K_3^{-1} \otimes F_3$  can be obtained by Corollary 2.1. The most important relations are the  $q$ -Serre relations. Since the generators  $E_i$ 's belong to  $(m^+)^i_{i+1}, i = 1, 2$ , we will consider the following equalities

$$\left. \begin{aligned} (m^+)_2^1 &= (q - q^{-1}) E_1 (m^+)_2^2, \\ e^3 (m^+)_2^1 &= \lambda R_{ab}^{13} (m^+)_2^a e^b = \lambda q (m^+)_2^1 e^3, \\ e^3 (m^+)_2^2 &= \lambda R_{ab}^{23} (m^+)_2^a e^b = \lambda q (m^+)_2^2 e^3. \end{aligned} \right\} \implies e^3 E_1 = E_1 e^3, \implies E_3 E_1 = E_1 E_3.$$

$$\left. \begin{aligned} (m^+)_3^2 &= (q - q^{-1})E_2(m^+)_3^3, \\ e^3(m^+)_3^2 &= \lambda R_{ab}^{23}(m^+)_3^a e^b = \lambda q(m^+)_3^2 e^3 + \lambda(q^2 - 1)(m^+)_3^3 e^2, \\ e^3(m^+)_3^3 &= \lambda R_{ab}^{33}(m^+)_2^a e^b = \lambda q^2(m^+)_3^3 e^3. \end{aligned} \right\} \implies e^2 = e^3 E_2 - q^{-1} E_2 e^3.$$

So, we need to explore the relation between  $e^2$  and  $e^3$ . Note that  $e^2 e^3 = R'_{ab}{}^{32} e^a e^b = q^{-2} R_{ab}^{32} e^a e^b = q^{-2} R_{32}^{32} e^3 e^2 = q^{-1} e^3 e^2$ , then combining with  $e^2 = e^3 E_2 - q^{-1} E_2 e^3$ , we obtain

$$(E_3)^2 E_2 - (q + q^{-1}) E_3 E_2 E_3 + E_2 (E_3)^2 = 0.$$

On the other hand, we need to know another relation between  $e^2$  and  $E_2$ , which can be explored by the following cross relations

$$\left. \begin{aligned} (m^+)_3^2 &= (q - q^{-1})E_2(m^+)_3^3, \\ e^2(m^+)_3^2 &= \lambda q^2(m^+)_3^2 e^2, \\ e^2(m^+)_3^3 &= \lambda q(m^+)_3^3 e^3, \\ e^2 &= e^3 E_2 - q^{-1} E_2 e^3. \end{aligned} \right\} \implies (E_2)^2 E_3 - (q + q^{-1}) E_2 E_3 E_2 + E_3 (E_2)^2 = 0.$$

The  $q$ -Serre relation of  $F_i$ ,  $1 \leq i \leq 3$  can be obtained similarly. Then the resulting quantum group is just the quantized enveloping algebra  $U_q(\mathfrak{sl}_4)$ , where  $e^i, f_i, i = 1, 2$  can be expressed by the  $q$ -commutators with generators  $E_i, F_i, i = 1, 2, 3$ .  $\square$

**Remark 3.1.** In the above concrete example, we find that it is not necessary to know the entire  $m^\pm$ -matrix for determining the structure of resulting quantum group, except for the diagonal and minor diagonal entries. At least, corresponding to the vector representation of  $U_q(\mathfrak{g})$ , all the simple root vectors  $E_i, F_i$  and group-like elements  $K_i$  are included in the diagonal and minor diagonal entries. Other entries in  $m^\pm$  are filled by non-simple root vectors generated by  $q$ -commutators with generators  $E_i, F_i$  and  $K_i$ . Moreover, the explicit expressions of the diagonal and minor diagonal entries in  $m^\pm$  can be easily obtained by Lemma 3.1.

For the  $A$  series, the data  $R, R', m^\pm$  as above are deduced from vector representation, which inspires us to start with the vector representation when considering the general rank-inductive construction. The  $R_{VV}$ -matrix of vector representation  $T_V$  satisfies the quadratic equation  $(PR_{VV} - q^{\frac{n-1}{n}} I)(PR_{VV} + q^{-\frac{n+1}{n}} I) = 0$ . So setting  $R = q^{\frac{n+1}{n}} R_{VV}, R' = q^{-2} R$ , then we have  $(PR + I)(PR' - I) = 0$ , and  $R_{kl}^{ij} = qq^{\delta_{ij}} \delta_{ik} \delta_{jl} + (q^2 - 1) \delta_{il} \delta_{jk} \theta(j - i)$ , where

$$\theta(k) = \begin{cases} 1 & k > 0, \\ 0 & k \leq 0. \end{cases}$$

On the other hand, according to Lemma 3.1, we obtain the following

**Lemma 3.2.** *Corresponding to the vector representation, the diagonal and minor diagonal entries in FRT-matrix  $m^\pm$  of  $U_q^{ext}(\mathfrak{sl}_n)$  are given by*

$$\begin{aligned} (m^+)_{i+1}^i &= (q - q^{-1}) E_i K_1^{-\frac{1}{n}} K_2^{-\frac{2}{n}} \cdots K_{i-1}^{-\frac{i-1}{n}} K_i^{-\frac{i}{n}} K_{i+1}^{\frac{n-(i+1)}{n}} \cdots K_{n-1}^{\frac{n-(n-1)}{n}}, \quad 1 \leq i \leq n-1, \\ (m^+)_{i+1}^i &= K_1^{-\frac{1}{n}} K_2^{-\frac{2}{n}} \cdots K_{i-1}^{-\frac{i-1}{n}} K_i^{\frac{n-i}{n}} \cdots K_{n-1}^{\frac{n-(n-1)}{n}}, \quad 1 \leq i \leq n. \\ (m^-)_{i+1}^{i+1} &= (q - q^{-1}) K_1^{\frac{1}{n}} K_2^{\frac{2}{n}} \cdots K_{i-1}^{\frac{i-1}{n}} K_i^{\frac{i}{n}} K_{i+1}^{-\frac{n-(i+1)}{n}} \cdots K_{n-1}^{-\frac{n-(n-1)}{n}} F_i, \quad 1 \leq i \leq n-1, \\ (m^-)_{i+1}^i &= K_1^{\frac{1}{n}} K_2^{\frac{2}{n}} \cdots K_{i-1}^{\frac{i-1}{n}} K_i^{-\frac{n-i}{n}} \cdots K_{n-1}^{-\frac{n-(n-1)}{n}}, \quad 1 \leq i \leq n. \end{aligned}$$

Obviously, we observe that  $(m^+)_{i+1}^i K_i^{-1} = (m^+)_{i+1}^{i+1}$  for  $1 \leq i \leq n-1$  by the above lemma. With these, we have the following

**Theorem 3.1.** *For type A, let  $\lambda = q^{-\frac{n+1}{n}}$ , and identify  $e^n, f_n, (m^+)_{i+1}^i c^{-1}$  with the additional simple root vectors  $E_n, F_n$  and group-like element  $K_n$ . Then the resulting quantum group  $U(V^\vee(R_{21}^{-1}, R'), U_q^{ext}(\mathfrak{sl}_n), V(R, R'))$  is exactly the  $U_q(\mathfrak{sl}_{n+1})$  with  $K_i^{\pm \frac{1}{n}}$  adjoined.*

*Proof.*  $(m^+)_{i+1}^i = K_1^{-\frac{1}{n}} K_2^{-\frac{2}{n}} \cdots K_{n-1}^{-\frac{n-1}{n}}$  can be obtained by Lemma 3.2.  $[E_n, F_n] = \frac{K_n - K_n^{-1}}{q - q^{-1}}$ ,  $\Delta(E_n) = E_n \otimes K_n + 1 \otimes E_n$ , and  $\Delta(F_n) = F_n \otimes 1 + K_n^{-1} \otimes F_n$  can be deduced easily from Corollary 2.1. On the other hand, we have the following cross relations by Corollary 2.1

$$\left. \begin{aligned} (m^-)_{i+1}^{i+1} &= (q - q^{-1})(m^-)_{i+1}^{i+1} F_i, \\ (m^-)_{i+1}^{i+1} e^n &= \lambda R_{n,i+1}^{n,i+1} e^n (m^-)_{i+1}^{i+1}, \\ (m^-)_{i+1}^{i+1} e^n &= \lambda R_{n,i+1}^{n,i+1} e^n (m^-)_{i+1}^{i+1}, \end{aligned} \right\} \implies F_i e^n = e^n F_i, \implies [E_n, F_i] = 0, \quad 1 \leq i \leq n-1.$$

$$\left. \begin{aligned} (m^+)_{i+1}^i &= (q - q^{-1}) E_i (m^+)_{i+1}^{i+1}, \\ (m^+)_{i+1}^i f_n &= \lambda f_n (m^+)_{i+1}^i R_{i+1,n}^{i+1,n}, \\ (m^+)_{i+1}^i f_n &= \lambda f_n (m^+)_{i+1}^i R_{i+1,n}^{i+1,n}, \end{aligned} \right\} \implies E_i f_n = f_n E_i, \implies [E_i, F_n] = 0, \quad 1 \leq i \leq n-1.$$

With these, we get

$$[E_n, F_i] = \delta_{ni} \frac{K_n - K_n^{-1}}{q - q^{-1}}, \quad [E_i, F_n] = \delta_{in} \frac{K_n - K_n^{-1}}{q - q^{-1}}. \quad (A1)$$

The relations between  $E_n, F_n$  and  $K_i$ ,  $1 \leq i \leq n$  also can be obtained by the cross relations in Corollary 2.1,

$$\left. \begin{aligned} (m^+)_{i+1}^i K_i^{-1} &= (m^+)_{i+1}^{i+1}, \quad 1 \leq i \leq n-1, \\ e^n (m^+)_{i+1}^i &= \lambda R_{i+1,n}^{i+1,n} (m^+)_{i+1}^i e^n, \\ R_{i+1,n}^{i+1,n} &= q, \quad 1 \leq i \leq n-1, \quad R_{nn}^{nn} = q^2. \end{aligned} \right\} \implies \begin{cases} E_n K_j = K_j E_n, & 1 \leq j \leq n-2, \\ E_n K_{n-1} = q^{-1} K_{n-1} E_n. \end{cases} \quad (A2)$$

$$\left. \begin{aligned} (m^+)_i K_i^{-1} &= (m^+)_{i+1}^{i+1}, \quad 1 \leq i \leq n-1, \\ (m^+)_i f_n &= \lambda f_n (m^+)_{i+1}^{i+1}, \\ R_{jn}^{jn} &= q, \quad 1 \leq j \leq n-1, \quad R_{nn}^{nn} = q^2. \end{aligned} \right\} \Rightarrow \begin{cases} F_n K_j = K_j F_n, \quad 1 \leq j \leq n-2, \\ F_n K_{n-1} = q K_{n-1} F_n. \end{cases} \quad (A3)$$

In the  $U_q(\mathfrak{sl}_n)$ ,  $E_i K_i = q^2 K_i E_i$ ,  $E_i K_{i\pm 1} = q^{-1} K_{i\pm 1} E_i$ ,  $E_i K_j = K_j E_i$ ,  $j \neq i \pm 1$ ;  $F_i K_i = q^{-2} K_i F_i$ ,  $F_i K_{i\pm 1} = q K_{i\pm 1} F_i$ ,  $F_i K_j = K_j F_i$ ,  $j \neq i \pm 1$ . Then we get the relations between  $K_n$  and  $E_i, F_i$ , where  $1 \leq i \leq n-2$ .

$$\begin{cases} E_i K_n = q^{\frac{i-1}{n}} q^{-\frac{2i}{n}} q^{\frac{i+1}{n}} K_1^{-\frac{1}{n}} \cdots K_{n-1}^{-\frac{n-1}{n}} c^{-1} E_i = K_n E_i, \\ F_i K_n = q^{-\frac{i-1}{n}} q^{\frac{2i}{n}} q^{-\frac{i+1}{n}} K_1^{-\frac{1}{n}} \cdots K_{n-1}^{-\frac{n-1}{n}} c^{-1} F_i = K_n F_i, \\ E_{n-1} K_n = q^{\frac{n-2}{n}} q^{-\frac{2(n-1)}{n}} K_1^{-\frac{1}{n}} \cdots K_{n-2}^{-\frac{n-2}{n}} K_{n-1}^{-\frac{n-1}{n}} c^{-1} E_{n-1} = q^{-1} K_n E_{n-1}, \\ F_{n-1} K_n = q^{-\frac{n-2}{n}} q^{\frac{2(n-1)}{n}} K_1^{-\frac{1}{n}} \cdots K_{n-2}^{-\frac{n-2}{n}} K_{n-1}^{-\frac{n-1}{n}} c^{-1} F_{n-1} = q K_n F_{n-1}. \end{cases} \quad (A4)$$

We want to explore the relations between  $e^n$  and  $E_i$ ,  $1 \leq i \leq n-1$ , and observe that  $E_i$  just belongs to the entry  $(m^+)_{i+1}^i$ , so

$$\left. \begin{aligned} (m^+)_{i+1}^i &= (q - q^{-1}) E_i (m^+)_{i+1}^{i+1}, \\ e^n (m^+)_{i+1}^i &= \lambda R_{in}^{in} (m^+)_{i+1}^i e^n, \quad 1 \leq i \leq n-2, \\ e^n (m^+)_{n-1}^{n-1} &= \lambda R_{n-1,n}^{n-1,n} (m^+)_{n-1}^{n-1} e^n + \lambda R_{n,n-1}^{n-1,n} (m^+)_{n-1}^{n-1} e^{n-1}, \\ e^n (m^+)_{i+1}^{i+1} &= \lambda R_{i+1,n}^{i+1,n} (m^+)_{i+1}^{i+1} e^n, \\ R_{jn}^{jn} &= q, \quad 1 \leq j \leq n-1, \quad R_{nn}^{nn} = q^2, \quad R_{n,n-1}^{n-1,n} = q^2 - 1. \end{aligned} \right\} \Rightarrow \begin{cases} E_n E_i = E_i E_n, \quad 1 \leq i \leq n-2, \\ e^{n-1} = e^n E_{n-1} - q^{-1} E_{n-1} e^n. \end{cases}$$

$$\left. \begin{aligned} (m^-)_i^{i+1} &= (q - q^{-1}) (m^-)_{i+1}^{i+1} F_i, \\ f_n (m^-)_i^{i+1} &= \lambda R_{ni}^{ni} (m^-)_i^{i+1} f_n, \quad 1 \leq i \leq n-2, \\ f_n (m^-)_{n-1}^n &= \lambda R_{n,n-1}^{n,n-1} (m^-)_{n-1}^n f_n + \lambda R_{n,n-1}^{n-1,n} (m^-)_{n-1}^n f_{n-1}, \\ f_n (m^-)_{i+1}^{i+1} &= \lambda R_{n,i+1}^{n,i+1} (m^-)_{i+1}^{i+1} f_n, \end{aligned} \right\} \Rightarrow \begin{cases} F_n F_i = F_i F_n, \quad 1 \leq i \leq n-2, \\ f_{n-1} = q f_n F_{n-1} - F_{n-1} f_n. \end{cases}$$

Note that  $e^{n-1} e^n = R_{ab}^{n,n-1} e^a e^b = q^{-2} R_{n,n-1}^{n,n-1} e^n e^{n-1} = q^{-1} e^n e^{n-1}$  and  $f_{n-1} f_n = f_b f_a R_{n-1,n}^{ab} = q^{-2} f_n f_{n-1} R_{n-1,n}^{n-1,n-1} = q^{-1} f_n f_{n-1}$ . Then we obtain

$$\begin{cases} (E_n)^2 E_{n-1} - (q + q^{-1}) E_n E_{n-1} E_n + E_{n-1} (E_n)^2 = 0, \\ (F_n)^2 F_{n-1} - (q + q^{-1}) F_n F_{n-1} F_n + F_{n-1} (F_n)^2 = 0. \end{cases} \quad (A5)$$

On the other hand,

$$\left. \begin{aligned} (m^+)_{n-1}^{n-1} &= (q - q^{-1}) E_{n-1} (m^+)_{n-1}^n, \\ e^{n-1} (m^+)_{n-1}^{n-1} &= \lambda q^2 (m^+)_{n-1}^{n-1} e^{n-1}, \\ e^{n-1} (m^+)_{n-1}^n &= \lambda q (m^+)_{n-1}^n e^{n-1}. \end{aligned} \right\} \Rightarrow e^{n-1} E_{n-1} = q E_{n-1} e^{n-1}.$$

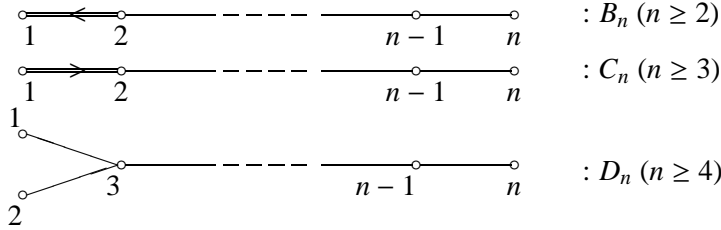
$$\left. \begin{aligned} (m^-)_{n-1}^n &= (q - q^{-1}) (m^-)_{n-1}^n F_{n-1}, \\ f_{n-1} (m^-)_{n-1}^n &= \lambda q^2 (m^-)_{n-1}^n f_{n-1}, \\ f_{n-1} (m^-)_{n-1}^n &= \lambda q (m^-)_{n-1}^n f_{n-1}. \end{aligned} \right\} \Rightarrow f_{n-1} F_{n-1} = q F_{n-1} f_{n-1}.$$

Combining with  $e^{n-1} = e^n E_{n-1} - q^{-1} E_{n-1} e^n$  and  $f_{n-1} = q f_n F_{n-1} - F_{n-1} f_n$ , we get

$$\begin{cases} (E_{n-1})^2 E_n - (q + q^{-1}) E_{n-1} E_n E_{n-1} + E_n (E_{n-1})^2 = 0, \\ (F_{n-1})^2 F_n - (q + q^{-1}) F_{n-1} F_n F_{n-1} + F_n (F_{n-1})^2 = 0. \end{cases} \quad (A6)$$

The other elements  $e^i, f_j$  can be identified with non-simple root vectors generated by  $q$ -commutators with  $E_i, F_i, K_i, 1 \leq i \leq n$ . With the above equalities (A1)—(A6), we prove that the resulting quantum groups is  $U_q(\mathfrak{sl}_{n+1})$ .  $\square$

**3.2. Inductive construction of  $U_q(\mathfrak{g})$  for the  $BCD$  series.** The rank-inductive construction of  $U_q(\mathfrak{sl}_n)$  gives us the confidence to consider the  $BCD$  series. Their Dynkin diagrams are given respectively by the following diagrams, and the arrow is point to the shorter of the two roots in the diagrams.



Corresponding to their vector representations, the matrix  $PR_{VV}$  satisfies the following cubic equation [FRT, KS]:

$$(PR_{VV} + q^{-1}I)(PR_{VV} - qI)(PR_{VV} - \epsilon q^{\epsilon-N}I) = 0.$$

Then set  $R = qR_{VV}$ ,  $R' = RPR - (\epsilon q^{\epsilon-N+1} + q^2)R + (\epsilon q^{\epsilon-N+3} + 1)P$ , we get  $(PR+I)(PR'-I) = 0$ . Moreover, there is a unique formula for the matrix entries of  $R$ ,

$$R_{kl}^{ij} = q q^{\delta_{ji} - \delta_{ji'}} \delta_{ik} \delta_{jl} + (q^2 - 1) \theta(j - l) (\delta_{il} \delta_{jk} - K_{lk}^{ij}).$$

Here  $K_{lk}^{ij} = \epsilon C_j^i C_k^l$ , and  $C_t^m = \epsilon_m \delta_{mt'} q^{-\rho_m}$ , where  $i' = N + 1 - i$ , let  $N = 2n$  if  $N$  is even, and  $N = 2n + 1$  if  $N$  is odd.  $\rho_i = \frac{N}{2} - i$  if  $i < i'$ ;  $\rho_{i'} = -\rho_i$  if  $i \leq i'$ ,  $\epsilon = \epsilon_1 = \dots = \epsilon_N = 1$  for  $g = \mathfrak{so}_N$ , and  $\rho_i = \frac{N}{2} + 1 - i$ ,  $\rho_{i'} = -\rho_i$  if  $i < i'$ ,  $\epsilon_1 = \dots = \epsilon_n = 1$ ,  $\epsilon = \epsilon_{n+1} = \dots = \epsilon_N = -1$  for  $g = \mathfrak{sp}_N$ .

For their vector representations, the diagonal and minor diagonal entries we need in  $m^\pm$  can be obtained by Lemma 3.1.

**Lemma 3.3.** (1) For  $U_q^{ext}(\mathfrak{so}_{2n+1})$  :

$$\begin{aligned} (m^+)_i^i &= K_1 K_2 \cdots K_{n-i} K_{n+1-i}, & (m^+)_{n+1}^{n+1} &= 1, & 1 \leq i \leq n, \\ (m^+)_{i+1}^i &= -(q - q^{-1}) E_{n+1-i} K_1 K_2 \cdots K_{n-i}, & (m^+)_{n+1}^n &= -c_0 E_1, \\ (m^-)_i^{i+1} &= (q - q^{-1}) K_1^{-1} K_2^{-1} \cdots K_{n-i}^{-1} F_{n+1-i}, & m_n^{-(n+1)} &= c_0 F_1, & (m^+)_i^i (m^-)_i^i &= 1, \end{aligned}$$

where  $c_0 = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{\frac{1}{2}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ .

(2) For  $U_q^{ext}(\mathfrak{sp}_{2n})$  :

$$\begin{aligned} (m^+)_i^i &= K_1^{\frac{1}{2}} K_2 \cdots K_{n+1-i}, & (m^+)_{i'}^{i'} (m^+)_i^i &= 1, & 1 \leq i \leq n, \\ (m^+)_{i+1}^i &= -(q-q^{-1})E_{n+1-i}K_1^{\frac{1}{2}}K_2 \cdots K_{n-i}, & 1 \leq i \leq n-1, & & (m^+)_{n+1}^n = -(q^2-q^{-2})E_1K_1^{-\frac{1}{2}}, \\ (m^-)_i^i &= K_1^{-\frac{1}{2}}K_2^{-1} \cdots K_{n+1-i}^{-1}, & (m^-)_{i'}^{i'} (m^-)_i^i &= 1, & 1 \leq i \leq n, \\ (m^-)_i^{i+1} &= (q-q^{-1})K_1^{-\frac{1}{2}}K_2^{-1} \cdots K_{n-i}^{-1}F_{n+1-i}, & 1 \leq i \leq n-1, & & (m^-)_n^{n+1} = (q^2-q^{-2})K_1^{\frac{1}{2}}F_1. \end{aligned}$$

(3) For  $U_q^{ext}(\mathfrak{so}_{2n})$  :

$$\begin{aligned} (m^+)_i^i &= (K_1^{\frac{1}{2}}K_2^{\frac{1}{2}})K_3 \cdots K_{n+1-i}, & (m^+)_{i'}^{i'} (m^+)_i^i &= 1, & 1 \leq i \leq n-2, \\ (m^+)_{n-1}^{n-1} &= K_1^{\frac{1}{2}}K_2^{\frac{1}{2}}, & (m^+)_n^n &= K_1^{\frac{1}{2}}K_2^{-\frac{1}{2}}, & (m^+)_{n+1}^{n-1} = -(q-q^{-1})E_1(K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}}), \\ (m^+)_{i+1}^i &= -(q-q^{-1})E_{n+1-i}(K_1^{\frac{1}{2}}K_2^{\frac{1}{2}})K_3 \cdots K_{n-i}, & 1 \leq i \leq n-1, \\ (m^-)_i^i &= (K_1^{-\frac{1}{2}}K_2^{-\frac{1}{2}})K_3^{-1} \cdots K_{n+1-i}^{-1}, & (m^-)_{i'}^{i'} (m^-)_i^i &= 1, & 1 \leq i \leq n-2, \\ (m^-)_{n-1}^{n-1} &= K_1^{-\frac{1}{2}}K_2^{-\frac{1}{2}}, & (m^-)_n^n &= K_1^{-\frac{1}{2}}K_2^{\frac{1}{2}}, & (m^-)_{n+1}^{n-1} = (q-q^{-1})(K_1^{\frac{1}{2}}K_2^{-\frac{1}{2}})F_1, \\ (m^-)_i^{i+1} &= (q-q^{-1})(K_1^{-\frac{1}{2}}K_2^{-\frac{1}{2}})K_3^{-1} \cdots K_{n-i}^{-1}F_{n+1-i}, & 1 \leq i \leq n-1. \end{aligned}$$

With these, we have the following

**Theorem 3.2.** With quantum group normalization constant  $\lambda = q^{-1}$ .

(1) Type B

Identify  $e^{2n+1}, f_{2n+1}, (m^+)_{2n+1}^{2n+1}c^{-1}$  with the additional simple root vectors  $E_{n+1}, F_{n+1}$  and the group-like element  $K_{n+1}$ . Then the resulting quantum group  $U(V^\vee(R', R_{21}^{-1}), U_q^{ext}(\widetilde{\mathfrak{so}_{2n+1}}), V(R', R))$  is the quantum group  $U_q(\mathfrak{so}_{2n+3})$ .

(2) Type C

Identify  $e^{2n}, f_{2n}, (m^+)_{2n}^{2n}c^{-1}$  with the additional simple root vectors  $E_{n+1}, F_{n+1}$  and the group-like element  $K_{n+1}$ . Then the resulting quantum group  $U(V^\vee(R', R_{21}^{-1}), U_q^{ext}(\widetilde{\mathfrak{sp}_{2n}}), V(R', R))$  is the quantum group  $U_q(\mathfrak{sp}_{2n+2})$  with  $K_1^{\pm\frac{1}{2}}$  adjoined.

(3) Type D

Identify  $e^{2n}, f_{2n}, (m^+)_{2n}^{2n}c^{-1}$  with the additional simple root vectors  $E_{n+1}, F_{n+1}$  and the group-like element  $K_{n+1}$ . Then the resulting quantum group  $U(V^\vee(R', R_{21}^{-1}), U_q^{ext}(\widetilde{\mathfrak{so}_{2n}}), V(R', R))$  is the quantum group  $U_q(\mathfrak{so}_{2n+2})$  with  $K_1^{\pm\frac{1}{2}}K_2^{\pm\frac{1}{2}}, K_1^{\pm\frac{1}{2}}K_2^{\mp\frac{1}{2}}$  adjoined.

*Proof.* The proof of Theorem 3.1 means that the relations of negative part can be obtained in a similar way, so we only focus on the relations of the positive part.

(1)  $(m^+)_{2n+1}^{2n+1}c^{-1} = K_1^{-1} \cdots K_{n-1}^{-1}K_n^{-1}$  follows from Lemma 3.3. From the identification in the above theorem, it is easily deduced from Corollary 2.1 that  $[E_{n+1}, F_{n+1}] = \frac{K_{n+1}-K_{n+1}^{-1}}{q-q^{-1}}$ ,  $\Delta(E_{n+1}) = E_{n+1} \otimes K_{n+1} + 1 \otimes E_{n+1}$ , and  $\Delta(F_{n+1}) = F_{n+1} \otimes 1 + K_{n+1}^{-1} \otimes F_{n+1}$ .

On the other hand, we have  $E_{n+1}K_{n+1} = e^{2n+1}(m^+)_{2n+1}^{2n+1}c^{-1} = \lambda R_a^{2n+1, 2n+1}(m^+)_{2n+1}^a e^b c^{-1} = R_{2n+1, 2n+1}^{2n+1}(m^+)_{2n+1}^{2n+1}c^{-1}e^{2n+1} = q^2 K_{n+1}E_{n+1}$ . The relations between the additional simple root vector  $e^{2n+1}$  and other  $K_i$ ,  $1 \leq i \leq n$  can be deduced from  $e^{2n+1}(m^+)_p^p = \lambda R_{ab}^{p, 2n+1}(m^+)_p^a e^b = \lambda R_{p, 2n+1}^{p, 2n+1}(m^+)_p^p e^{2n+1}$ ,  $1 \leq p \leq n+1$ . Combining with  $(m^+)_{i+1}^{i+1}K_{n+1-i} = (m^+)_i^i$ ,  $1 \leq i \leq n$ , and  $R_{12n+1}^{12n+1} = 1$ ,  $R_{p2n+1}^{p2n+1} = q$ ,  $2 \leq p \leq n+1$ , we obtain  $e^{2n+1}K_n = q^{-1}K_n e^{2n+1}$ ,  $e^{2n+1}K_i = K_i e^{2n+1}$ , for  $1 \leq i \leq n-1$ , namely,

$$E_{n+1}K_n = q^{-1}K_n E_{n+1}, \quad E_{n+1}K_i = K_i E_{n+1}, \quad 1 \leq i \leq n-1.$$

We observe that  $F_i$  belongs to  $(m^-)_i^{i+1}$ , so the relations between  $E_{n+1}$  and  $F_i$  can be obtained by the equality  $(m^-)_j^i e^k = \lambda R_a^k{}_b^i e^a (m^-)_j^b$  in Corollary 2.1.

$$\left. \begin{aligned} (m^-)_i^{i+1} &= (q - q^{-1})(m^-)_{i+1}^{i+1}F_i, \quad 1 \leq i \leq n-1, \\ (m^-)_i^{i+1}e^{2n+1} &= \lambda R_a^{2n+1, i+1}e^a (m^-)_i^b = e^{2n+1}(m^-)_i^{i+1}, \quad 1 \leq i \leq n, \\ (m^-)_p^p e^{2n+1} &= \lambda R_a^{2n+1, p}e^a (m^-)_p^b = e^{2n+1}(m^-)_p^p, \quad 2 \leq p \leq n, \end{aligned} \right\} \implies [E_{n+1}, F_i] = 0, \quad 1 \leq i \leq n.$$

We will explore the  $q$ -Serre relations of the positive part. We also observe that  $E_{n+1-i}$  belongs to  $(m^+)_{i+1}^i$ ,  $1 \leq i \leq n$ , so

$$\left\{ \begin{aligned} e^{2n+1}(m^+)_2^1 &= \lambda R_{ab}^{1, 2n+1}(m^+)_2^a e^b = \lambda R_{12n+1}^{1, 2n+1}(m^+)_2^1 e^{2n+1} + \lambda R_{22n}^{1, 2n+1}(m^+)_2^2 e^{2n}, \\ e^{2n+1}(m^+)_{i+1}^i &= \lambda R_{ab}^{i, 2n+1}(m^+)_{i+1}^a e^b = \lambda R_{i2n+1}^{i, 2n+1}(m^+)_{i+1}^i e^{2n+1} = \lambda q(m^+)_{i+1}^i e^{2n+1}, \quad 2 \leq i \leq n. \end{aligned} \right.$$

Putting the expression of  $(m^+)_{i+1}^i$ ,  $1 \leq i \leq n$  into the above equalities, we get

$$\left\{ \begin{aligned} e^{2n+1}E_j &= E_j e^{2n+1}, \quad 1 \leq j \leq n-1, \\ e^{2n} &= e^{2n+1}E_n - q^{-1}E_n e^{2n+1}. \end{aligned} \right.$$

So, we need to know the relations between  $e^{2n}$  and  $e^{2n+1}, E_n$ . We have  $e^{2n+1}e^{2n} = R_a^{2n, 2n+1}e^a e^b = -(q + \epsilon q^{\epsilon-N+2})e^{2n}e^{2n+1} + (\epsilon q^{\epsilon-N+1} + 2)e^{2n+1}e^{2n}$ , so  $e^{2n+1}e^{2n} = qe^{2n}e^{2n+1}$ . Combining with  $e^{2n} = e^{2n+1}E_n - q^{-1}E_n e^{2n+1}$ , we get  $(e^{2n+1})^2 E_n - (q + q^{-1})e^{2n+1}E_n e^{2n+1} + E_n(e^{2n+1})^2 = 0$ , namely,

$$(E_{n+1})^2 E_n - (q + q^{-1})E_{n+1}E_n E_{n+1} + E_n(E_{n+1})^2 = 0.$$



On the other hand, according to the equality  $e^{2n}(m^+)_2^1 = \lambda R_{ab}^{12n}(m^+)_2^a e^b = \lambda R_{12n}^{12n}(m^+)_2^1 e^{2n} = (m^+)_2^1 e^{2n}$  and  $e^{2n}(m^+)_2^2 = \lambda R_{22n}^{22n}(m^+)_2^2 e^{2n} = q^{-1}(m^+)_2^2 e^{2n}$ , we get  $e^{2n}E_n = qE_n e^{2n}$ . Combining with  $e^{2n} = e^{2n+1}E_n - q^{-1}E_n e^{2n+1}$  again, we obtain

$$(E_n)^2 E_{n+1} - (q + q^{-1})E_n E_{n+1} E_n + E_{n+1}(E_n)^2 = 0.$$

With these relations, we prove that the resulting quantum group is  $U_q(\mathfrak{so}_{2n+3})$ . (2) and (3) can be proved in a similar way.  $\square$

#### 4. TYPE-CROSSING CONSTRUCTIONS OF TYPES $BCD$ STARTING FROM TYPE $A$

Up to now, we have got the general inductive constructions of the classical quantum groups within the same type as in the above section. However, Majid claimed that his double-bosonization allows to create not only a line of nodes diagram but also a tree of nodes diagram of quantum groups. At each node of the tree, we have more choices to adjoin suitable braided groups to obtain possible different new quantum groups of higher rank one. In this section, we will give some examples to demonstrate this fact. As we known, at the source node corresponding to  $U_q(\mathfrak{sl}_2)$ , Majid [M3] chose a pair of braided groups generated by the vector representation of  $U_q(\mathfrak{sl}_2)$  to give  $U_q(\mathfrak{sl}_3)$  as above. In what follows, we will give 3 kinds of type-crossing constructions: from type  $A_1$  to type  $B_2$ , from type  $A_2$  to type  $C_3$ , and from type  $A_3$  to type  $D_4$ .

Corresponding to an irreducible representation of  $U_q(\mathfrak{g})$ , the matrix  $R_{VV}$  is induced by  $R_{VV} = B_{VV} \circ (T_V \otimes T_V)(\mathfrak{R})$ . Here  $\mathfrak{R} = \sum_{r_1, \dots, r_n=0}^{\infty} \prod_{j=1}^n \frac{(1-q_{\beta_j}^{-2})^{r_j}}{[r_j]_{q_{\beta_j}}!} q_{\beta_j}^{\frac{r_j(r_j+1)}{2}} E_{\beta_j}^{r_j} \otimes F_{\beta_j}^{r_j}$  is the main part of the universal  $R$ -matrix of  $U_h(\mathfrak{g})$ .  $B_{VV}$  denotes the linear operator on  $V \otimes V$  given by  $B_{VV}(v \otimes w) := q^{(\mu, \mu')} v \otimes w$  for  $v \in V_{\mu}$ ,  $w \in V_{\mu'}$ .

**4.1. Type-crossing construction from type  $A_1$  to type  $B_2$ .** Now, we still start from the node diagram of  $U_q(\mathfrak{sl}_2)$  and choose other braided groups to give  $U_q(\mathfrak{so}_5)$  in the following.

**Example 4.1.** Starting from a 3-dimensional representation  $T_V$  of  $U_q(\mathfrak{sl}_2)$ , which is given by  $E_1(x_i) = [2]_q x_{i+1}$ ,  $E_1(x_3) = 0$ ,  $F_1(x_{i+1}) = x_i$ ,  $F_1(x_1) = 0$ ,  $i = 1, 2$ , where  $x_1, x_2, x_3$  is the

base of  $V$  with corresponding weights  $-\alpha_1, 0, \alpha_1$ . Then we get a  $9 \times 9$   $R_{VV}$ -matrix datum

$$R_{VV} = \begin{pmatrix} q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & q^2 - q^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-2} & 0 & q^2 - q^{-2} & 0 & (1 - q^{-2})(q^2 - q^{-2}) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 - q^{-4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & q^2 - q^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 \end{pmatrix}.$$

Clearly,  $R_{VV}$  is invertible, and according to the submodules decomposition of module  $V^{\otimes 2}$ , it is easy to see that  $PR_{VV}$  obeys the minimal polynomial

$$(PR_{VV} + q^{-2}I)(PR_{VV} - q^2I)(PR_{VV} - q^{-4}I) = 0.$$

Setting  $R = q^2 R_{VV}$ ,  $R' = RPR - q^{-2}R - q^4R + (q^2 + 1)P$ , then

$$(PR + I)(PR' - I) = 0.$$

With these  $R$  and  $R'$ , choose a pair of braided groups  $V^\vee(R', R_{21}^{-1})$ ,  $V(R', R)$ , and  $\lambda = q^{-2}$ . Identify  $e^3, f_3, (m^+)_3 c^{-1}$  with the additional simple root vectors  $E_2, F_2$  and the group-like element  $K_2$ . Then the resulting quantum group  $U(V^\vee(R', R_{21}^{-1}), U_q^{\text{ext}}(\mathfrak{sl}_2), V(R', R))$  is the quantum group  $U_q(\mathfrak{so}_5)$ .

*Proof.* Corresponding to this representation, we get the  $m^\pm$ -matrix as follows

$$m^+ = \begin{pmatrix} K_1 & -(q - q^{-1})E_1 & \frac{q(1 - q^{-2})^2}{[2]_q} E_1^2 K_1^{-1} \\ 0 & 1 & -(q - q^{-1})E_1 K_1^{-1} \\ 0 & 0 & K_1^{-1} \end{pmatrix},$$

$$m^- = \begin{pmatrix} K_1^{-1} & 0 & 0 \\ (q^4 - 1)F_1 & 1 & 0 \\ (q^4 - q^2)(q^4 - 1)K_1 F_1^2 & (q^2 - q^{-2})K_1 F_1 & K_1 \end{pmatrix}.$$

With the identification as above,  $[E_2, F_2] = \frac{K_2 - K_2^{-1}}{q^2 - q^{-2}}$ ,  $\Delta(E_2) = E_2 \otimes K_2 + 1 \otimes E_2$ , and  $\Delta(F_2) = F_2 \otimes 1 + K_2^{-1} \otimes F_2$  can be deduced from Corollary 2.1.

$E_2 K_2 = e^3 (m^+)_3 c^{-1} = \lambda R_{33}^{33} (m^+)_3 e^3 c^{-1} = R_{33}^{33} (m^+)_3 c^{-1} e^3 = q^4 K_2 E_2$ ,  $E_2 K_1 = q^{-2} K_1 E_2$  can be deduced from  $e^3 (m^+)_1^1 = \lambda R_{13}^{13} (m^+)_1^1 e^3 = q^{-2} (m^+)_1^1 e^3$ . On the other hand,  $E_1 K_2 = E_1 (m^+)_3 c^{-1} = E_1 K_1^{-1} c^{-1} = q^{-2} K_1^{-1} c^{-1} E_1 = q^{-2} K_2 E_1$ . According to the equality  $(m^-)_1^2 e^3 =$

$\lambda R_{32}^{32} e^3 (m^-)_1^2 = e^3 (m^-)_1^2$ , we obtain  $e^3 F_1 = F_1 e^3$  combining with  $(m^-)_1^2 = (q^4 - 1)F_1$ , namely,  $[E_2, F_1] = 0$ .

We will explore the  $q$ -Serre relation between  $e^3$  and  $E_1$ . The equality  $(q + q^{-1})e^2 = q^{-2}E_1 e^3 - e^3 E_1$  can be given by  $e^3 (m^+)_2^1 = \lambda R_{ab}^{13} (m^+)_2^a e^b = q^{-2} (m^+)_2^1 e^3 + (q^2 - q^{-2}) (m^+)_2^2 e^2$ . Combining with  $e^3 e^2 = q^2 e^2 e^3$ , which is deduced from  $e^2 e^3 = R'_{ab}{}^{32} e^a e^b = (q^4 + q^2 + 1) e^2 e^3 - (q^2 + 1) e^3 e^2$ , we obtain

$$(E_2)^2 E_1 - (q^2 + q^{-2}) E_2 E_1 E_2 + E_1 (E_2)^2 = 0.$$

On the other hand, we need to know the relation between  $e^2$  and  $E_1$ .

$$\left. \begin{aligned} e^2 (m^+)_2^1 &= \lambda R_{ab}^{12} (m^+)_2^a e^b = (m^+)_2^1 e^2 + (q^2 - q^{-2}) (m^+)_2^2 e^1, \\ e^1 (m^+)_2^1 &= \lambda R_{ab}^{11} (m^+)_2^a e^b = \lambda R_{11}^{11} (m^+)_2^1 e^1 = q^2 (m^+)_2^1 e^1. \end{aligned} \right\} \Rightarrow \begin{cases} e^1 = \frac{1}{q+q^{-1}} (E_1 e^2 - e^2 E_1), \\ e^1 E_1 = q^2 E_1 e^1. \end{cases}$$

Combining with  $(q + q^{-1})e^2 = q^{-2}E_1 e^3 - e^3 E_1$ , we obtain

$$(E_1)^3 E_2 - \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q (E_1)^2 E_2 E_1 + \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q E_1 E_2 (E_1)^2 - E_2 (E_1)^3 = 0.$$

With these relations, the Cartan matrix of the resulting quantum group is  $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ .

So the resulting quantum group is  $U_q(\mathfrak{so}_5)$ , and the proof is complete.  $\square$

**4.2. Type-crossing construction from type  $A_2$  to type  $C_3$ .** In Example 3.1, we get  $U_q(\mathfrak{sl}_4)$  starting from  $U_q(\mathfrak{sl}_3)$  by choosing the braided groups generated by a 3-dimensional vector representation via the Majid's double-bosonization construction. In the example below, we will choose another pair of braided groups generated by a 6-dimensional irreducible module to give  $U_q(\mathfrak{sp}_6)$  starting from the node diagram of  $U_q(\mathfrak{sl}_3)$ .

**Example 4.2.** *The pair of braided groups we want to have is obtained from the 6-dimensional irreducible representation  $T_V$  of  $U_q(\mathfrak{sl}_3)$ , which is defined by*

$$\begin{aligned} E_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} x_2 \\ (q + q^{-1})x_4 \\ x_5 \end{pmatrix}, & E_2 \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} &= \begin{pmatrix} x_3 \\ x_5 \\ (q + q^{-1})x_6 \end{pmatrix}, \\ F_1 \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} &= \begin{pmatrix} (q + q^{-1})x_1 \\ x_2 \\ x_3 \end{pmatrix}, & F_2 \begin{pmatrix} x_3 \\ x_5 \\ x_6 \end{pmatrix} &= \begin{pmatrix} x_2 \\ (q + q^{-1})x_4 \\ x_5 \end{pmatrix}, \end{aligned}$$

where  $\{x_i \mid 1 \leq i \leq 6\}$  is a basis of  $V$  with the corresponding weights  $-2\lambda_1, -2\lambda_1 + \alpha_1, -2\lambda_1 + \alpha_1 + \alpha_2, -2\lambda_1 + 2\alpha_1, -2\lambda_1 + 2\alpha_1 + \alpha_2, -2\lambda_1 + 2\alpha_1 + 2\alpha_2$ , respectively. We can

obtain a  $36 \times 36$  matrix  $R_{VV}$  corresponding to the 6-dimensional representation, and the  $PR_{VV}$  obeys the minimal polynomial

$$(PR_{VV} - q^{\frac{8}{3}}I)(PR_{VV} + q^{-\frac{4}{3}}I)(PR_{VV} - q^{-\frac{10}{3}}I) = 0.$$

Setting  $R = q^{\frac{4}{3}}R_{VV}$ ,  $R' = RPR - (q^{-2} + q^4)R + (q^2 + 1)P$ , we have

$$(PR + I)(PR' - I) = 0,$$

and  $\lambda = q^{-\frac{4}{3}}$ . Identify  $e^6, f_6, (m^+)^6 c^{-1}$  with the additional simple root vectors  $E_3, F_3$  and the group-like element  $K_3$ , then the resulting quantum group  $U(V^\vee(R', R_{21}^{-1}), U_q^{ext}(\widehat{\mathfrak{sl}_3}), V(R', R))$  is the quantum group  $U_q(\mathfrak{sp}_6)$  with  $K_i^{\frac{1}{3}}$  ( $i = 1, 2$ ) adjoined.

*Proof.* The elements in matrix  $m^\pm$  we need can be obtained by Lemma 3.1, which are given by the following equalities

$$\left\{ \begin{array}{l} (m^+)_2^1 = -(q - q^{-1})E_1 K_1^{\frac{1}{3}} K_2^{\frac{2}{3}}, \quad (m^+)_2^2 = K_1^{\frac{1}{3}} K_2^{\frac{2}{3}}, \\ (m^+)_6^5 = -(q^2 - q^{-2})E_2 K_1^{-\frac{2}{3}} K_2^{-\frac{4}{3}}, \quad (m^+)_6^6 = K_1^{-\frac{2}{3}} K_2^{-\frac{4}{3}}, \quad (m^+)_4^4 = K_1^{-\frac{2}{3}} K_2^{\frac{2}{3}}, \\ (m^-)_3^5 = q(q - q^{-1})K_1^{\frac{2}{3}} K_2^{\frac{1}{3}} F_1, \quad (m^-)_5^5 = K_1^{\frac{2}{3}} K_2^{\frac{1}{3}}, \\ (m^-)_5^6 = (q - q^{-1})K_1^{\frac{2}{3}} K_2^{\frac{4}{3}} F_2, \quad (m^-)_6^6 = K_1^{\frac{2}{3}} K_2^{\frac{4}{3}}. \end{array} \right.$$

We only focus on the relations of the positive part.

Note that  $E_3 K_3 = e^6 (m^+)_6^6 c^{-1} = \lambda R_{66}^{66} (m^+)_6^6 e^6 c^{-1} = R_{66}^{66} (m^+)_6^6 c^{-1} e^6 = q^4 (m^+)_6^6 c^{-1} e^6 = q^4 K_3 E_3$ . Combining with  $e^6 (m^+)_i^i = \lambda R_{i6}^{i6} (m^+)_i^i e^6$ , we have

$$\left\{ \begin{array}{l} (m^+)_6^6 K_2^2 = (m^+)_4^4, \\ e^6 (m^+)_6^6 = q^{\frac{8}{3}} (m^+)_6^6 e^6, \\ e^6 (m^+)_4^4 = q^{-\frac{4}{3}} (m^+)_4^4 e^6. \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} e^6 K_1 = K_1 e^6, \\ e^6 K_2 = q^{-2} K_2 e^6. \end{array} \right. \Rightarrow \left\{ \begin{array}{l} E_3 K_1 = K_1 E_3, \\ E_3 K_2 = q^{-2} K_2 E_3. \end{array} \right.$$

On the other hand,  $E_2 K_3 = E_2 K_1^{-\frac{2}{3}} K_2^{-\frac{4}{3}} c^{-1} = q^{\frac{2}{3}} q^{-\frac{8}{3}} K_1^{-\frac{2}{3}} K_2^{-\frac{4}{3}} c^{-1} E_2 = q^{-2} K_3 E_2$ ,  $E_1 K_3 = E_1 K_1^{-\frac{2}{3}} K_2^{-\frac{4}{3}} c^{-1} = q^{-\frac{4}{3}} q^{\frac{4}{3}} K_1^{-\frac{2}{3}} K_2^{-\frac{4}{3}} c^{-1} E_1 = K_3 E_1$ . We will explore the  $q$ -Serre relations. Since  $E_1, E_2$  belong to  $(m^+)_2^1, (m^+)_6^5$ , respectively, then

$$\left\{ \begin{array}{l} (m^+)_2^1 = -(q - q^{-1})E_1 (m^+)_2^2, \\ e^6 (m^+)_2^1 = q^{-\frac{4}{3}} (m^+)_2^1 e^6, \\ e^6 (m^+)_2^2 = q^{-\frac{4}{3}} (m^+)_2^2 e^6. \end{array} \right\} \Rightarrow e^6 E_1 = E_1 e^6, \Rightarrow E_3 E_1 = E_1 E_3.$$

$$\left\{ \begin{array}{l} (m^+)_6^5 = -(q^2 - q^{-2})E_2 (m^+)_6^6, \\ e^6 (m^+)_6^5 = q^{\frac{2}{3}} (m^+)_6^5 e^6 + (q^2 - q^{-2})q^{\frac{2}{3}} (m^+)_6^6 e^5, \\ e^6 (m^+)_6^6 = q^{\frac{8}{3}} (m^+)_6^6 e^6. \end{array} \right\} \Rightarrow e^5 = q^{-\frac{4}{3}} (q^{-2} E_2 e^6 - e^6 E_2).$$

So we need to know the relation between  $e^5$  and  $e^6$ .  $e^6e^5 = q^2e^5e^6$  is obtained by  $e^ie^j = R'_{ab}{}^{ji}e^ae^b$  and  $R'_{56}{}^{56} = -q^2 - 1$ ,  $R'_{65}{}^{56} = 2 + q^{-2}$ , then we have

$$(E_3)^2E_2 - (q^2 + q^{-2})E_3E_2E_3 + E_2(E_3)^2.$$

On the other hand, we need to obtain the relation between  $e^5$  and  $E_2$ .

$$\left. \begin{aligned} (m^+)_6^5 &= -(q^2 - q^{-2})E_2(m^+)_6^6, \\ e^5(m^+)_6^5 &= q^{\frac{2}{3}}(m^+)_6^5e^5 + (q - q^{-1})q^{-\frac{4}{3}}(m^+)_6^6e^4, \\ e^4(m^+)_6^6 &= q^{-\frac{4}{3}}(m^+)_6^6e^4. \end{aligned} \right\} \implies e^4 = -(q + q^{-1})(E_2e^5 - e^5E_2).$$

Then the relation between  $e^5$  and  $E_2$  must be deduced from the relation between  $e^4$  and  $E_2$ , which is given by the following cross relations

$$\left. \begin{aligned} (m^+)_6^5 &= -(q^2 - q^{-2})E_2(m^+)_6^6, \\ e^4(m^+)_6^5 &= q^{\frac{2}{3}}(m^+)_6^5e^4, \\ e^4(m^+)_6^6 &= q^{-\frac{4}{3}}(m^+)_6^6e^4. \end{aligned} \right\} \implies e^4E_2 = q^2E_2e^4 \implies (E_2e^5 - e^5E_2)E_2 = q^2E_2(E_2e^5 - e^5E_2).$$

Combining with  $e^5 = q^{-\frac{4}{3}}(q^{-2}E_2e^6 - e^6E_2)$ , we obtain  $(E_2)^3e^6 - (q^2 + 1 + q^{-2})(E_2)^2e^6E_2 + (q^2 + 1 + q^{-2})E_2e^6(E_2)^2 - e^6(E_2)^3 = 0$ , namely,

$$(E_2)^3E_3 - \left[ \begin{matrix} 3 \\ 1 \end{matrix} \right]_q (E_2)^2E_3E_2 + \left[ \begin{matrix} 3 \\ 2 \end{matrix} \right]_q E_2E_3(E_2)^2 - E_3(E_2)^3 = 0.$$

From these relations, it follows that the resulting quantum group is  $U_q(\mathfrak{sp}_6)$ .  $\square$

**4.3. Type-crossing construction from type  $A_3$  to type  $D_4$ .** In the following example, we can choose the different braided groups generated by a 6-dimensional  $U_q(\mathfrak{sl}_4)$ -module to give  $U_q(\mathfrak{so}_8)$  based on the node diagram of  $U_q(\mathfrak{sl}_4)$ .

**Example 4.3.** *There is a 6-dimensional irreducible representation  $T_V$  of  $U_q(\mathfrak{sl}_4)$ , given by*

$$\begin{aligned} E_1 \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} x_4 \\ x_5 \end{pmatrix}, & E_2 \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} &= \begin{pmatrix} x_2 \\ x_6 \end{pmatrix}, & E_3 \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} &= \begin{pmatrix} x_3 \\ x_5 \end{pmatrix}; \\ F_1 \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} &= \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}, & F_2 \begin{pmatrix} x_2 \\ x_6 \end{pmatrix} &= \begin{pmatrix} x_1 \\ x_5 \end{pmatrix}, & F_3 \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} &= \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}. \end{aligned}$$

Here  $\{x_i \mid 1 \leq i \leq 6\}$  is a basis of  $V$  with corresponding weights  $-2\lambda_1 + \alpha_1, -2\lambda_1 + \alpha_1 + \alpha_2, -2\lambda_1 + \alpha_1 + \alpha_2 + \alpha_3, -2\lambda_1 + 2\alpha_1 + \alpha_2, -2\lambda_1 + 2\alpha_1 + \alpha_2 + \alpha_3, -2\lambda_1 + 2\alpha_1 + 2\alpha_2 + \alpha_3$ , respectively. Then we get another  $36 \times 36$  matrix  $R_{VV}$ , and the  $PR_{VV}$  matrix obeys the minimal polynomial

$$(PR_{VV} + q^{-1}I)(PR_{VV} - q^{-1}I)(PR_{VV} - qI) = 0.$$

Setting  $R = qR_{VV}$ ,  $R' = RPR - (q^2 + 1)R + (q^2 + 1)P$ , then we have

$$(PR + I)(PR' - I) = 0,$$

and  $\lambda = q^{-1}$ . Identify  $e^6, f_6, (m^+)_6 c^{-1}$  with the additional simple root vectors  $E_4, F_4$  and the group-like element  $K_4$ . Then the resulting quantum group  $U(V^\vee(R', R_{21}^{-1}), U_q^{\text{ext}}(\mathfrak{sl}_4), V(R', R))$  is the quantum group  $U_q(\mathfrak{so}_8)$  with  $K_i^{\frac{1}{2}}, 1 \leq i \leq 3$  adjoined.

*Proof.* The entries in matrix  $m^\pm$  we need are listed by the following equalities

$$\left\{ \begin{array}{l} (m^+)_2^1 = -(q - q^{-1})E_2 K_1^{\frac{1}{2}} K_3^{\frac{1}{2}}, \quad (m^+)_2^2 = K_1^{\frac{1}{2}} K_3^{\frac{1}{2}}, \\ (m^+)_3^2 = -(q - q^{-1})E_3 K_1^{\frac{1}{2}} K_3^{-\frac{1}{2}}, \quad (m^+)_3^3 = K_1^{\frac{1}{2}} K_3^{-\frac{1}{2}}, \\ (m^+)_4^2 = -(q - q^{-1})E_1 K_1^{-\frac{1}{2}} K_3^{\frac{1}{2}}, \quad (m^+)_4^4 = K_1^{-\frac{1}{2}} K_3^{\frac{1}{2}}, \quad (m^+)_6^6 = K_1^{-\frac{1}{2}} K_2^{-1} K_3^{-\frac{1}{2}}, \\ (m^-)_1^2 = q(q - q^{-1})K_1^{-\frac{1}{2}} K_3^{-\frac{1}{2}} F_2, \quad (m^-)_2^2 = K_1^{-\frac{1}{2}} K_3^{-\frac{1}{2}}, \\ (m^-)_2^3 = q(q - q^{-1})K_1^{-\frac{1}{2}} K_3^{\frac{1}{2}} F_3, \quad (m^-)_3^3 = K_1^{-\frac{1}{2}} K_3^{\frac{1}{2}}, \\ (m^-)_3^5 = q(q - q^{-1})K_1^{\frac{1}{2}} K_3^{\frac{1}{2}} F_1, \quad (m^-)_5^5 = K_1^{\frac{1}{2}} K_3^{\frac{1}{2}}. \end{array} \right.$$

The cross relations can be easily obtained as above, so we only describe the  $q$ -Serre relations of the positive part.  $E_1, E_3$  belong to  $(m^+)_4^2, (m^+)_3^2$ , respectively, so we have

$$\left\{ \begin{array}{l} (m^+)_4^2 = -(q - q^{-1})E_1(m^+)_4^4, \\ e^6(m^+)_4^2 = \lambda R_{26}^{26}(m^+)_4^2 e^6 = (m^+)_4^2 e^6, \\ e^6(m^+)_4^4 = (m^+)_4^4 e^6. \end{array} \right\} \implies e^6 E_1 = E_1 e^6, \implies E_4 E_1 = E_1 E_4.$$

$$\left\{ \begin{array}{l} (m^+)_3^2 = -(q - q^{-1})E_3(m^+)_3^3, \\ e^6(m^+)_3^2 = \lambda R_{26}^{26}(m^+)_3^2 e^6 = (m^+)_3^2 e^6, \\ e^6(m^+)_3^3 = (m^+)_3^3 e^6. \end{array} \right\} \implies e^6 E_3 = E_3 e^6, \implies E_4 E_3 = E_3 E_4.$$

$E_2$  belongs to  $(m^+)_2^1$ , then we obtain

$$\left\{ \begin{array}{l} (m^+)_2^1 = -(q - q^{-1})E_2(m^+)_2^2, \\ e^6(m^+)_2^1 = q^{-1}(m^+)_2^1 e^6 + q^{-1}(q - q^{-1})(m^+)_2^2 e^5, \\ e^6(m^+)_2^2 = (m^+)_2^2 e^6, \quad e^5(m^+)_2^2 = q^{-1}(m^+)_2^2 e^5. \end{array} \right\} \implies e^5 = q^{-1}E_2 e^6 - e^6 E_2.$$

Combining with  $e^6 e^5 = q e^5 e^6$ , which is obtained by  $e^i e^j = R'^{ji}_{ab} e^a e^b$  and  $R'^{56}_{56} = -2q, R'^{56}_{65} = 3$ , we have

$$E_2(E_4)^2 - (q + q^{-1})E_4 E_2 E_4 + (E_4)^2 E_2 = 0.$$

On the other hand, combining with  $e^5 E_2 = q E_2 e^5$  deduced from  $e^5(m^+)_2^1 = \lambda R_{15}^{15}(m^+)_2^1 e^5$ , we obtain  $E_4(E_2)^2 - (q + q^{-1})E_2 E_4 E_2 + (E_2)^2 E_4 = 0$ .  $\square$

**Remark 4.1.** *Observing the constructions of  $C_3$  and  $D_4$ , we claim that  $C_{n+1}$  and  $D_{n+1}$  can be constructed directly from the node diagram  $A_n$ . The pairs of braided groups or the  $R$ -matrices data  $R, R'$  we should choose will be obtained by the ‘symmetric square’ and the second exterior power of the vector representation of  $A_n$ . The verification of the claim will rely on some skills, full details of it will be developed in a sequel.*

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